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On the Sharpness of a Jackson Estimate by Ditzian–Totik

L. Imhof

Lehrstuhl A für Mathematik, RWTH Aachen, D-52056 Aachen, Germany

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Using ideas of Grünwald, Marcinkiewicz, and Vértesi concerning the divergence of interpolation processes, a counterexample is constructed which establishes that a Jackson estimate for the best approximation by algebraic polynomials given by Ditzian and Totik is sharp in a pointwise sense everywhere.

1. INTRODUCTION

Let $C[-1,1]$ be the space of functions f, continuous on the interval $[-1,1]$, endowed with the uniform norm $\|f\| := \max_{-1\leqslant x\leqslant 1} |f(x)|$, and for $n\in\mathbb{N}$ (set of natural numbers) let $\mathscr{E}_n(f;x)$ be the best approximation to f from the set $\mathscr{P}_n$ of algebraic polynomials of degree at most n, thus

$$\|\mathscr{E}_n(f;x)-f(x)\| = \min_{p\in\mathscr{P}_n} \|p(x)-f(x)\|.$$

Following Z. Ditzian and V. Totik [5] we define the weighted modulus of smoothness of order $r\in\mathbb{N}$ by means of

$$\omega_\varphi^r(f;t) := \sup_{0\leqslant h\leqslant t} \|\Delta_{h\varphi(x)}^r f(x)\|, \qquad \varphi(x) := (1-x^2)^{1/2},$$

$$\Delta_h^r f(x) := \begin{cases} \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x+rh/2-kh), & \text{if} \quad |x\pm rh/2|\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

In these terms one has for $f\in C[-1,1]$ (see [5, p. 79])

$$|\mathscr{E}_n(f;x)-f(x)| \leqslant C\omega_\varphi^r\left(f;\frac{1}{n}\right), \qquad x\in[-1,1], \qquad n>r, \qquad (1.1)$$

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where $C = C(r)$ depends on r only. To investigate the sharpness of this Jackson estimate, let us consider a function $\omega_r(t)$ with the following properties (cf. [7]):

$$0 = \omega_r(0) < \omega_r(t) \leqslant \omega_r(T) \qquad \text{if} \quad 0 < t \leqslant T,$$

$$\omega_r(t) \text{ is continuous for } \quad t \geqslant 0, \tag{1.2}$$

$$T^r/\omega_r(T) \geqslant t^r/\omega_r(t) \qquad \text{if} \quad T \geqslant t > 0, \tag{1.3}$$

$$\lim_{t \to 0+} t^r/\omega_r(t) = 0. \tag{1.4}$$

From (1.1) it immediately follows that

$$\omega_\varphi^r(f; t) = \mathcal{O}(\omega_r(t)) \quad \Rightarrow \quad |\mathscr{E}_n(f; x) - f(x)| = \mathcal{O}\left(\omega_r\left(\frac{1}{n}\right)\right).$$

This is best possible in the sense that, under appropriate conditions, even a converse result holds true (see [2, p. 265] for the case $\omega_r(t) = t^\alpha$, $0 < \alpha < r$, see also [4]). It is also sharp in the following sense.

THEOREM. *For each $\omega_r(t)$ satisfying (1.2–4) there exists a counterexample $f \in C[-1, 1]$ such that $\omega_\varphi^r(f; t) = \mathcal{O}(\omega_r(t))$ and*

$$\limsup_{n \to \infty} \frac{|\mathscr{E}_n(f; x) - f(x)|}{\omega_r(1/n)} \geqslant c > 0 \tag{1.5}$$

simultaneously for each $x \in [-1, 1]$.

A proof of this assertion will be given in Section 3, whereas two Lemmas, stated and proved in the next section, will pave the way. Let us point out that in the above situation a counterexample $f \in C[-1, 1]$ satisfying $\omega_\varphi^r(f; t) = \mathcal{O}(\omega_r(t))$ and

$$\limsup_{n \to \infty} \|\mathscr{E}_n(f; x) - f(x)\|/\omega_r(1/n) \geqslant c > 0$$

can be obtained by means of an uniform boundedness principle with rates ([3]) and the equivalence of ω_φ^r with a weighted K-functional (see [2, p. 187 ff.]). To examine the behaviour of the pointwise (rather than norm) error $|\mathscr{E}_n(f; x) - f(x)|$, some ideas of [6; 7] turn out to be useful.

2. AUXILIARY RESULTS

Let T_n denote the Chebyshev polynomial of degree n, thus $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$. In order to prove the theorem we investigate certain linear combinations of these polynomials.

LEMMA 1. *For* $0 < 8\mu \leqslant \lambda$, $n \in \mathbb{N}$ *let* $g \in C[-1, 1]$ *be defined by*

$$g(x) := \lambda T_{n+1}(x) + \mu T_{2(n+1)}(x), \qquad x \in [-1, 1].$$

For $m = n+1, 2(n+1)$ *consider the sets* $\mathbb{B}_m := \{x \in [-1, 1] : |T_m(x)| \geqslant 1/2\}$. *Then*

$$|\mathscr{E}_n(g; x) - g(x)| \geqslant \frac{\lambda}{4} \qquad \textit{for} \quad x \in \mathbb{B}_{n+1}, \tag{2.1}$$

$$|\mathscr{E}_{2n+1}(g; x) - g(x)| \geqslant \frac{\mu}{2} \qquad \textit{for} \quad x \in \mathbb{B}_{2(n+1)}, \tag{2.2}$$

$$\mathbb{B}_{n+1} \cup \mathbb{B}_{2(n+1)} = [-1, 1]. \tag{2.3}$$

Proof. With $x = \cos\theta$, $\theta \in (0, \pi)$, one easily calculates

$$\begin{aligned} g'(x) &= (n+1)\,\lambda\,\frac{\sin\,(n+1)\,\theta}{\sin\theta} + 2(n+1)\,\mu\,\frac{\sin\,2(n+1)\,\theta}{\sin\theta} \\ &= (n+1)\,\frac{\sin\,(n+1)\,\theta}{\sin\theta}\,[\lambda + 4\mu\cos(n+1)\,\theta]. \end{aligned}$$

Since $\lambda + 4\mu\cos(n+1)\,\theta \geqslant \lambda - 4\mu > 0$, one obtains

$$\begin{aligned} g'(x) > 0 \quad &\Leftrightarrow \quad \theta \in \bigcup_{\substack{k=0 \\ k \text{ even}}}^{n} \left(\frac{k\pi}{n+1}, \frac{(k+1)\,\pi}{n+1}\right) \\ &\Leftrightarrow \quad x \in \bigcup_{\substack{k=0 \\ k \text{ even}}}^{n} (\eta_{k+1}^{(n+1)}, \eta_k^{(n+1)}), \\ g'(x) < 0 \quad &\Leftrightarrow \quad \theta \in \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{n} \left(\frac{k\pi}{n+1}, \frac{(k+1)\,\pi}{n+1}\right) \\ &\Leftrightarrow \quad x \in \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{n} (\eta_{k+1}^{(n+1)}, \eta_k^{(n+1)}), \end{aligned}$$

where $\eta_k^{(n+1)} := \cos(k\pi/(n+1))$, $k = 0, 1, ..., n+1$. Therefore the function $g(x) - \mu$ alternately attains the maximum λ (at $x = \eta_k^{(n+1)}$, k even) and the minimum $-\lambda$ (at $x = \eta_k^{(n+1)}$, k odd); and there are no further extrema in $[-1, 1]$. That is, $\eta_{n+1}^{(n+1)} < \eta_n^{(n+1)} < \cdots < \eta_0^{(n+1)}$ form an alternating set for $g(x) - \mu$. In this situation the Chebyshev theorem ([2, p. 74]) says that μ is the best approximation to g out of $\mathscr{P}_n$, whence

$$|\mathscr{E}_n(g; x) - g(x)| = |\mu - g(x)| = |\mu - \lambda T_{n+1}(x) - \mu T_{2(n+1)}(x)|$$
$$\geqslant \lambda\, |T_{n+1}(x)| - 2\mu \geqslant \lambda \left(|T_{n+1}(x)| - \frac{1}{4} \right) \geqslant \frac{\lambda}{4}$$

for $x \in \mathbb{B}_{n+1}$. Thus (2.1) is established.

Another application of Chebyshev's theorem delivers $\mathscr{E}_{2n+1}(\mu T_{2(n+1)}; x) \equiv 0$. Therefore by virtue of $\mathscr{E}_{2n+1}(f + p) = \mathscr{E}_{2n+1}(f) + p$ for all $f \in C[-1, 1]$, $p \in \mathscr{P}_{2n+1}$ (see [1, p. 83])

$$|\mathscr{E}_{2n+1}(g; x) - g(x)| = |\mathscr{E}_{2n+1}(\mu T_{2(n+1)}; x) - \mu T_{2(n+1)}(x)| = \mu\, |T_{2(n+1)}(x)|,$$

and (2.2) again follows from the definition of $\mathbb{B}_{2(n+1)}$.

If $x = \cos\theta \notin \mathbb{B}_{n+1}$, then $|\cos(n+1)\,\theta| < 1/2$. In view of $\cos 2\vartheta = 2\cos^2\vartheta - 1$ it follows that $|T_{2(n+1)}(x)| = |\cos 2(n+1)\,\theta| \geqslant 1/2$, hence $x \in \mathbb{B}_{2(n+1)}$. This proves (2.3). ∎

The following is merely a reformulation of an inequality due to Freud (see [2, p. 77]).

Lemma 2. *For each $g \in C[-1, 1]$ and $n \in \mathbb{N}$ exists a constant $K = K(g, n) > 1$ such that for every $h \in C[-1, 1]$*

$$|\mathscr{E}_n(h; x) - h(x)| \geqslant |\mathscr{E}_n(g; x) - g(x)| - K\, \|h - g\|, \qquad x \in [-1, 1].$$

Proof. According to the inequality mentioned there is a constant $\tilde{K} > 0$ such that $\|\mathscr{E}_n h - \mathscr{E}_n g\| \leqslant \tilde{K}\, \|h - g\|$ for all $h \in C[-1, 1]$. With $K := \tilde{K} + 1$ one obtains for $x \in [-1, 1]$

$$|\mathscr{E}_n(h; x) - h(x)| \geqslant |\mathscr{E}_n(g; x) - g(x)| - |\mathscr{E}_n(h; x) - \mathscr{E}_n(g; x)| - |h(x) - g(x)|$$
$$\geqslant |\mathscr{E}_n(g; x) - g(x)| - K\, \|h - g\|. \quad ∎$$

3. PROOF OF THE THEOREM

For each $n \in \mathbb{N}$ consider the function

$$g_n(x) := \omega_r\left(\frac{1}{n}\right) T_{n+1}(x) + \frac{1}{8}\, \omega_r\left(\frac{1}{2n+1}\right) T_{2(n+1)}(x) \in C[-1, 1].$$

According to Lemma 2 there are constants $K_n := K(g_n, n) + K(g_n, 2n+1) > 2$ such that for every $h \in C[-1, 1]$, $x \in [-1, 1]$, and $v \in \{n, 2n+1\}$

$$|\mathscr{E}_v(h; x) - h(x)| \geqslant |\mathscr{E}_v(g_n; x) - g_n(x)| - K_n \|h - g_n\|. \tag{3.1}$$

Starting with an arbitrary $n_1 \geqslant 2r$, one may choose a subsequence $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that for $k \geqslant 1$:

$$n_{k+1} > 2(n_k + 1), \tag{3.2}$$

$$\omega_r\left(\frac{1}{n_{k+1}}\right) \leqslant \frac{1}{64 K_{n_k}} \omega_r\left(\frac{1}{2n_k + 1}\right), \tag{3.3}$$

$$n_{k+1}^r \omega_r\left(\frac{1}{n_{k+1}}\right) \geqslant \sum_{j=1}^{k} n_j^r \omega_r\left(\frac{1}{n_j}\right). \tag{3.4}$$

By (1.2) and (3.3) one obtains $\omega_r(1/n_{j+1}) \leqslant \omega_r(1/(2n_j+1))/128 \leqslant \omega_r(1/n_j)/128$, hence

$$\sum_{j=k}^{\infty} \left[\omega_r\left(\frac{1}{n_j}\right) + \frac{1}{8} \omega_r\left(\frac{1}{2n_j + 1}\right)\right] \leqslant \frac{9}{8} \sum_{j=k}^{\infty} \omega_r\left(\frac{1}{n_j}\right) \leqslant 2\omega_r\left(\frac{1}{n_k}\right). \tag{3.5}$$

Since $\|T_n\| = 1$ for all $n \in \mathbb{N}$, this implies that $f(x) := \sum_{j=1}^{\infty} g_{n_j}(x)$ is well-defined in $C[-1, 1]$. To estimate $\omega_\varphi^r(f; t)$ we first show the following Jackson–Bernstein-type inequality:

$$\omega_\varphi^r(g_n; t) \leqslant M_r \omega_r\left(\frac{1}{n}\right) \min\{1, t^r n^r\}, \qquad 0 \leqslant t \leqslant \frac{1}{2r}, \qquad n \in \mathbb{N}. \tag{3.6}$$

Indeed, in view of the definition of the modulus of smoothness

$$\omega_\varphi^r(g_n; t) \leqslant 2^r \|g_n\| \leqslant 2^r \left(\omega_r\left(\frac{1}{n}\right) + \frac{1}{8} \omega_r\left(\frac{1}{2n+1}\right)\right) \leqslant 2^{r+1} \omega_r\left(\frac{1}{n}\right).$$

Moreover, using an estimate for the weighted modulus ([2, p. 187]) and a Bernstein-type inequality ([5, p. 107]) one has for $t \leqslant 1/2r$

$$\begin{aligned} \omega_\varphi^r(g_n; t) &\leqslant C_r t^r \|(1 - x^2)^{r/2} g_n^{(r)}(x)\| \\ &\leqslant \widetilde{C}_r\, t^r (2(n+1))^r \|g_n(x)\| \leqslant \widetilde{C}_r\, 2^{2r+1} t^r n^r \omega_r\left(\frac{1}{n}\right), \end{aligned}$$

and (3.6) is proven.

Now we fix $0 < t < 1/n_1$. Then there exists $k \in \mathbb{N}$ with $1/n_{k+1} \leqslant t < 1/n_k$, and by (1.2–3, 3.4–6) it follows that

$$\omega_\varphi^r(f;t) \leqslant \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty}\right) \omega_\varphi^r(g_{n_j};t)$$

$$\leqslant M_r t^r \sum_{j=1}^{k} n_j^r \omega_r\left(\frac{1}{n_j}\right) + M_r \sum_{j=k+1}^{\infty} \omega_r\left(\frac{1}{n_j}\right)$$

$$\leqslant 2M_r t^r n_k^r \omega_r\left(\frac{1}{n_k}\right) + 2M_r \omega_r\left(\frac{1}{n_{k+1}}\right) \leqslant 4M_r \omega_r(t),$$

whence $\omega_\varphi^r(f;t) = \mathcal{O}(\omega_r(t))$.

To obtain (1.5) we shall prove the following: For every $x \in [-1, 1]$ there is a subsequence $(v_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$|\mathscr{E}_{v_k}(f;x) - f(x)| \geqslant \frac{1}{32}\, \omega_r\left(\frac{1}{v_k}\right) \tag{3.7}$$

for all $k \in \mathbb{N}$. To this end, we fix $x^* \in [-1, 1]$ and, as in Lemma 1, define $\mathbb{B}_n := \{x \in [-1, 1] \colon |T_n(x)| \geqslant 1/2\}$, $n \in \mathbb{N}$. Then for each $k \in \mathbb{N}$ we choose the index v_k as follows: If $x^* \in \mathbb{B}_{n_k+1}$ let $v_k := n_k$, otherwise let $v_k := 2n_k + 1$. By (3.2) one has $v_1 < v_2 < \cdots$ and $g_{n_j} \in \mathscr{P}_{2(n_j+1)} \subset \mathscr{P}_{n_k} \subset \mathscr{P}_{v_k}$ for $1 \leqslant j \leqslant k-1$. Setting $f = (\sum_{j=1}^{k-1} + \sum_{j=k}^{\infty})\, g_{n_j} =: p + h$ and observing that $p \in \mathscr{P}_{v_k}$ and thus $\mathscr{E}_{v_k}(p+h) = p + \mathscr{E}_{v_k}(h)$ (see [1, p. 83]) one arrives at (cf. (3.1))

$$|\mathscr{E}_{v_k}(f;x^*) - f(x^*)| = |\mathscr{E}_{v_k}(h;x^*) - h(x^*)|$$

$$\geqslant |\mathscr{E}_{v_k}(g_{n_k};x^*) - g_{n_k}(x^*)| - K_{n_k}\, \|h - g_{n_k}\|. \tag{3.8}$$

By (1.2, 3.3, 3.5) one has

$$\|h - g_{n_k}\| \leqslant \sum_{j=k+1}^{\infty} \left[\omega_r\left(\frac{1}{n_j}\right) + \frac{1}{8}\, \omega_r\left(\frac{1}{2n_j+1}\right)\right]$$

$$\leqslant 2\omega_r\left(\frac{1}{n_{k+1}}\right) \leqslant \frac{1}{32K_{n_k}}\, \omega_r\left(\frac{1}{2n_k+1}\right) \leqslant \frac{1}{32K_{n_k}}\, \omega_r\left(\frac{1}{v_k}\right). \tag{3.9}$$

Consider the case $v_k = n_k$, i.e., $x^* \in \mathbb{B}_{n_k+1}$. Using Lemma 1 for $n = v_k$, $\lambda = \omega_r(1/n_k)$, $\mu = \omega_r(1/(2n_k+1))/8$, thus $g = g_{n_k}$, one obtains

$$|\mathscr{E}_{v_k}(g_{n_k};x^*) - g_{n_k}(x^*)| \geqslant \frac{1}{4}\, \omega_r\left(\frac{1}{n_k}\right) = \frac{1}{4}\, \omega_r\left(\frac{1}{v_k}\right). \tag{3.10}$$

If $v_k = 2n_k + 1$, i.e., $x^* \notin \mathbb{B}_{n_k+1}$, then one has in view of (2.3) that $x^* \in \mathbb{B}_{2(n_k+1)}$, and another application of Lemma 1 yields

$$|\mathscr{E}_{v_k}(g_{n_k}; x^*) - g_{n_k}(x^*)| \geqslant \frac{1}{16}\,\omega_r\left(\frac{1}{2n_k+1}\right) = \frac{1}{16}\,\omega_r\left(\frac{1}{v_k}\right). \tag{3.11}$$

Summarizing (3.8–11) one obtains (3.7) in any case, and therefore the proof is complete.

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